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ON GRADIENT RICCI SOLITONS AND YAMABE SOLITONS

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Abstract. In this paper, we consider gradient Ricci solitons and gradient Yamabe solitons in the warped product spaces. Also we study warped product space with harmonic curvature related to gradient Ricci solitons and gradient Yamabe solitons. Consequently some theorems are generalized and we derive differential equations for a warped product space to be a gradient Ricci soliton.

1. Introduction

A Riemannian manifold (M, g) with a Riemannian metric g is said to have a harmonic curvature [2] if the formal divergence δR of curvature tensor R vanishes. It is easily see that M has a harmonic curvature if and only if $(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0$ for all vector fields X, Y, and Z on M, where S is the Ricci curvature of M.

A Riemannian metric g on a complete Riemannian manifold M is called a Ricci soliton if there exists a smooth vector field X such that S satisfies the following equation

$$S + \frac{1}{2}\mathfrak{L}_X g = \rho g \tag{1.1}$$

for some constant ρ, where $\mathfrak{L}_X$ is the Lie derivative with respect to X [1,3,4,6,7,9,11,12]. It is well known that Ricci solitons are self-similar solitons to the Ricci flow which is introduced by R.S. Hamilton [5].

The Ricci soliton is called shrinking if $\rho > 0$, steady if $\rho = 0$ and expanding if $\rho < 0$. If $X = \nabla h$ for some smooth function h on M, then

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M is called a gradient Ricci soliton with (h, ρ) [11]. In this case, h is called a potential function and the equation (1.1) can be rewritten as

$$S + Hess\ h = \rho g. \tag{1.2}$$

It is well known that all compact steady or expanding solitons are necessarily Einstein [4], and a Ricci soliton on a compact manifold has a constant curvature in 2-dimension [5] as well as in 3-dimension [6]. Moreover a Ricci soliton on a compact manifold is a gradient Ricci soliton [7], and a compact shrinking soliton is always gradient [10].

On the other hand, in the noncompact case, Perelman [10] has studied and classified the 3-dimensional shrinking gradient Ricci solitons with bounded nonnegative sectional curvature.

Since Ricci solitons are natural extension of Einstein metrics, it is meaningful to construct a non-Einstein gradient Ricci soliton. In this point of view, we study warped product spaces with gradient Ricci solitons and consider the converse problem for the construction of non-Einstein gradient Ricci solitons.

A Riemannian metric g is called a Yamabe soliton if there exist a smooth vector field X and a constant ρ satisfying $(r - \rho)g = \frac{1}{2}\mathfrak{L}_X g$, where r is a scalar curvature of M. If a Riemannian manifold has a constant scalar curvature, then g becomes a trivial Yamabe soliton. Consequently we obtain some theorems generalizing the known results for the warped product space with harmonic curvature and induce a differential equation for a warped product space to be a gradient Ricci soliton.

2. Harmonic curvature in warped product spaces $M = R \times_f F$

It is well known that [2].

THEOREM 2.1. *A Riemannian manifold of dimension n has a harmonic curvature if and only if the scalar curvature r is constant and $D = 0$, where*

$$D(X,Y)Z = \frac{\{(\nabla_Y S)(X,Z) - (\nabla_X S)(Y,Z)\}}{n-2} + \frac{\{(Xr)g(Y,Z) - (Yr)g(X,Z)\}}{2(n-1)(n-2)},$$

which is conformally invariant in 3-dimensional case.

Let F be an n-dimensional Riemannian manifold with Riemannian metric $\bar{g}$. Then the product manifold $R \times F$ with a Riemannian metric $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & f^2\bar{g} \end{pmatrix}$ for a positive function f is called a warped product space of R and F with a warping function f. We denote it by $R \times_f F$.

In the warped product space $M = R\times_f F$, the Ricci curvature tensors $\tilde{S}$ and $\bar{S}$ of M and F are respectively given by [8, 9]

$$(2.1)\qquad \begin{aligned} \tilde{S}_{xy} &= \bar{S}_{xy} - ff_{11}\bar{g}_{xy} - (n-1)f_1^2\bar{g}_{xy}, \\ \tilde{S}_{x1} &= 0, \\ \tilde{S}_{11} &= -\tfrac{nf_{11}}{f} \\ \tilde{r} &= \tfrac{\bar{r}}{f^2} - \tfrac{2nf_{11}}{f} - \tfrac{n(n-1)f_1^2}{f^2}, \end{aligned}$$

where $f_1 = \frac{df}{dt}, f_{11} = \frac{d^2f}{dt^2}$, the range of indices $x, y, z, \cdots$, is $\{2, 3, \cdots, n+1\}$ and t is a variable on R.

Since the Riemannian manifold with a constant scalar curvature is a trivial Yamabe soliton, by use of Theorem 2.1 and the fourth equation of (2.1), we have

THEOREM 2.2. *If $M = R\times_f F$ has a harmonic curvature, then the scalar curvatures of M and F are constant. Hence M and F become trivial Yamabe solution.*

Proof. Since $\tilde{r} = \frac{\bar{r}}{f^2} - \frac{2nf_{11}}{f} - \frac{n(n-1)f_1^2}{f^2}$ and $\tilde{r}$ is constant by Theorem 2.1 and that $\partial_x\bar{r} = 0$, we see that $\bar{r}$ is constant on F.

THEOREM 2.3. *If $M = R\times_f F$ has a harmonic curvature with $n > 2$, then*
(1) F becomes Einstein if f is not constant.
(2) F has a harmonic curvature if f is constant.

Proof. From the equation (2.1), we obtain $0 = \tilde{\nabla}_x\tilde{S}_{1y} = \tilde{\nabla}_1\tilde{S}_{xy} = ((1-n)\partial_1||f_1||^2 - \partial_1(f\triangle f))\bar{g}_{xy} - \frac{2f_1}{f}(\bar{S}_{xy} - ff_{11}\bar{g}_{xy} - (n-1)f_1^2\bar{g}_{xy})$, where $\tilde{\nabla}$ and $\bar{\nabla}$ are covariant derivative operators on M and F respectively, and $\triangle$ is a Laplacian operator on R. Then we get $\bar{S}_{xy} = \frac{f}{2f_1}((1-n)\partial_1||f_1||^2 - \partial_1(f\triangle f) + ff_{11} - (n-1)f_1^2)\bar{g}_{xy} = A\bar{g}_{xy}$ for $A = \frac{f}{2f_1}((1-n)\partial_1||f_1||^2 - \partial_1(f\triangle f) + ff_{11} - (n-1)f_1^2)$. Then we see that $\partial_x A = 0$, where $n > 2$. Hence F becomes Einstein if $f_1 \neq 0$.

On the other hand, if $f_1 = 0$, then $\tilde{S}_{xy} = S_{xy}$ and that $\tilde{\nabla}_z\tilde{S}_{xy} = \bar{\nabla}_x\bar{S}_{zy}$, we see that F has a harmonic curvature.

3. Gradient Ricci soliton in $M = R \times_f F$

If $M = R \times_f F$ is a gradient Ricci soliton with (h, ρ), then we have

$$(3.1)\qquad \begin{aligned} \bar{S}_{xy} &= (ff_{11} + (n-1){f_1}^2 + \rho f^2 - \bar{\nabla}_x h_y - ff_1 h_1)\bar{g}_{xy}, \\ \partial_1 h_x &= \frac{f_1}{f} h_x, \\ \nabla_1 \nabla_1 h &= \rho + \frac{nf_{11}}{f}, \end{aligned}$$

where $h_1 = \partial_t h$ and $h_x = \partial_x h$.

By use of the equation (3.1), we have

Theorem 3.1. *Let $M = R \times_f F$ be a gradient Ricci soliton with (h, ρ). If h_y is a non-zero function without component of R for a certain variable y in $x_i (2 \le i \le n+1)$, then F is a gradient Ricci soliton and M is a Riemannian product of R and F.*

Proof. From the second equation of (3.1), we have $\partial_1 (ln \frac{h_y}{f}) = 0$, so $ln \frac{h_y}{f}$ depends only on F. Hence we can put $ln \frac{h_y}{f} = A(x_i)$ for a certain function A of $x_i (2 \le i \le n-1)$ which are variables of F. Therefore $h_y = fe^{A(x_i)}$ and we can express $h_y = fC(x_i)$ for $C(x_i) = e^{A(x_i)}$. Consequently h is of the form $h = f(L(x_i) + U(b, \hat{y}))$,where L is the partial integration of C with respect to y and $U(t, \hat{y})$ is a function on $R \times F$ with $\partial_y U = 0$. Successively we have $h_y = fC_y(x_i)$ and $\partial_1 h_y = f_1 C_y(x_i)$. The function $C_y(x_i) \neq 0$ because $h_y \neq 0$ for some y, so we obtain $f_1 = 0$. This means that M is a Riemannian products of R and F. Then, from the equation of (3.1), we have $\partial_1 h_x = 0$, $\nabla_1 \nabla_1 h = \rho$ and $\bar{S}_{xy} = \rho f^2 \bar{g}_{xy}$. Then we can put $h = P(x_i) + Q(t)$ for some functions P and Q on F and R respectively. So we get $\nabla_1 h_1 = \nabla_1 Q_1 = \rho$ and that F is a gradient Ricci soliton.

Theorem 3.2. *Let $M = R \times_f F$ be a gradient Ricci soliton with (h, ρ). If $h_1 = 0$, then F is a gradient Ricci soliton and the warping function f is given by $f = c_1 cos\mu t + c_2 sin\mu t$.*

Proof. Since $h_1 = 0$, the potential function h depends only on F. From the first and third equation of (3.1), we see that $\bar{S}_{xy} = A\bar{g}_{xy} - \bar{\nabla}_x h_y$ where $A = ff_{11} + (n-1)f_1^2 + \rho f^2 - \bar{\nabla}_x h_y - ff_1 h_1$ and $f_{11} + \frac{\rho}{n} f = 0$ respectively. Since the function A does not depend on F, F becomes a gradient Ricci soliton. Moreover we see that the general solution of $f_{11} + \frac{\rho}{n} f = 0$ is given by $f(t) = c_1 cos\mu t + c_2 sin\mu t$ for constants c_1 and c_2, where we have put $\mu = (\frac{\rho}{n})^{\frac{1}{2}}$.

From the first equation of (3.1), we see that F becomes Einstein if $h_y = 0$ for all y. Thus we have

THEOREM 3.3. *Let $M = R \times_f F$ be a gradient Ricci soliton with (h, ρ). If $h_y = 0$ for all y, then F becomes an Einstein space.*

For the converse of Theorem 3.3, if we suppose that F is Einstein, then $\bar{S}_{xy} = k\bar{g}_{xy}$ for the constant $k = \frac{\bar{r}}{n}$. Moreover we can calculate $\tilde{\nabla}_1 h_1 = h_{11}$,$\tilde{\nabla}_1 h_x = \partial_1 h_x - \frac{f_1}{f} h_x$, $\tilde{\nabla}_y h_x = \bar{\nabla}_y h_x + ff^1 h_1 \bar{g}_{yx}$, where h is a function on $R \times F$. Hence if we consider the equation (2.1) and the definition of gradient Ricci soliton (1.2), then we have

THEOREM 3.4. *Let F be an Einstein space with an Einstein constant k and let f be a positive function on R. If there exist some function h depends only on R and constant ρ satisfying*

$$\begin{aligned} \frac{nf_{11}}{f} &= h_{11} - \rho, \\ k - ff_{11} - (n-1)f_1^2 &= \rho f^2 - ff^1 h_1, \end{aligned} \tag{3.2}$$

then the warped product space $R \times_f F$ becomes a gradient Ricci soliton with (h, ρ).

The upper Theorem 3.4 provides the method of a construction for non-Einstein gradient Ricci soliton, in the warped product space $M = R \times_f F$. In the warped product space, if the warping function is not constant, then it is called that the warped product space is essential.

THEOREM 3.5. *Let $M = R \times_f F$ be an essential warped product space with gradient Ricci soliton with (h, ρ). If $h_1 \neq 0$, then F is an Einstein space.*

Proof. From the third equation of (3.1), we see that $\nabla_1 \nabla_1 h$ only depends on R. Since $\nabla_1 \nabla_1 h = \frac{\partial^2 h}{\partial t^2}$, we can put $\frac{\partial^2 h}{\partial t^2} = k(t)$. Then we get $h = K(t) + P(x)t + Q(x)$ from which $h_y = P_y(x)t + Q_y(x)$. From this fact and the second equation of (3.1), we have $\frac{f_1}{f} h_y = \partial_1 h_y = P_y(x)$. Since $f_1 \neq 0$, $h_y = \frac{f}{f_1} P_y(x)$. That is $P_y(x)t + Q_y(x) = \frac{f}{f_1} P(x)$. If we differentiate upper equation with respect to t, then we get $P_y(x) = \frac{\partial}{\partial t}(\frac{f}{f_1}) P_y(x)$. Hence we get $f_{11} P_y(x) = 0$, from which $ff_{11} = 0$ or $P_y(x) = 0$. But in the case of $ff_{11} = 0$, $f_1 = constant(c)$ and consequently $f = ct + d$. Since f is a positive function, f becomes a positive constant function. This is a contradition to $f_1 \neq 0$. Hence $P_y(x) = 0$ and that $h_y = 0$. Then we see that F becomes an Einstein space due to Theorem 3.3.

4. Gradient Ricci soliton in $M = B \times_f F$

Let (B, g) and $(F, \bar{g})$ be n and p dimensional Riemannian manifolds with Riemannian metrics g and $\bar{g}$ respectively. Then the warped product space $M = B \times_f F$ is the Riemannian manifold with Riemannian metric $\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & f^2\bar{g} \end{pmatrix}$ and warping function f.

If the warped product space $M = B \times_f F$ is a gradient Ricci soliton with (h, ρ), then we get [8]

$$\begin{aligned} S_{ab} &= \rho g_{ab} - \nabla_a h_b + \tfrac{p}{f}\nabla_a f_b, \\ \partial_a h_x &= \tfrac{f_a}{f} h_x, \\ \bar{S}_{xy} &= (\rho f^2 + f\triangle f + (p-1)||f_e||^2 - f f^a h_a)\bar{g}_{xy} - \bar{\nabla}_x\bar{\nabla}_y h, \end{aligned} \tag{4.1}$$

where the ranges of indices $a, b, \cdots$ and $x, y, \cdots$ are $\{1, 2, \cdots, n\}$ and $\{n+1, \cdots, n+p\}$ respectively.

In [9], the authors proved that

THEOREM 4.1. *Let $M = B \times_f F$ be a gradient Ricci soliton with (h, ρ). If $h_y \neq 0$ all y, then B becomes an Einstein space and $\nabla_b\nabla_a f = 0$ for all a and b, where ∇ is the covariant derivative operator on B.*

For the generalization of Theorem 4.1, we obtain

THEOREM 4.2. *Let $M = B \times_f F$ be a gradient Ricci soliton with (h, ρ). If $h_y \neq 0$ for some y, then B becomes a gradient Ricci soliton and $\nabla_b\nabla_a f = 0$ for all a and b.*

Proof. Suppose that $h_y \neq 0$ for some y. Then from the second equation of (4.1), we see that $\partial_a(ln\frac{h_y}{f}) = 0$ for all a. Hence we get $h_y = fe^{V(z)} = fC(z)$ for $C(z) = e^{V(z)}$, and that h is the form $h = f(P(z) + W(b, \hat{y}))$,where $P_y = C$ and $V(b, \hat{y})$ is a function on $B \times R$ with $\partial_y W = 0$. From this form, we get $h_x = f(P_x(z) + W_x(b, \hat{y}))$, and $\partial_a h_x = f_a(P_x(z) + W_x(b, \hat{y})) + fW_{xa}(b, \hat{y})$. By use of these equations and the second equation of (4.1), we have $W_{xa}(b, \hat{y}) = 0$ for all a and b. Henceforth we can express $W(b, \hat{y}) = E(c) + F(z, \hat{y})$. Then we get $h = f[P(z) + E(c) + F(z, \hat{y})]$. Hence we obtain

$$\nabla_a h_b = \nabla_a\nabla_b f[P(z)+E(c)+F(z,\hat{y})]+f_b E_a(c)+f_a E_b(c)+f\nabla_a\nabla_b E(c). \tag{4.2}$$

By use of the first equation of (4.1) and equation (4.2), we have $0 = \partial_y(\nabla_a h_b) = (\nabla_b\nabla_a f)P_y(z) = (\nabla_b\nabla_a f)C(z)$ because $P_y = C$. Therefore $\nabla_a\nabla_b f = 0$, because if $C(z) = 0$, then $h_y = 0$ which is a contradiction for the assumption. From the equations (4.1) and (4.2), we get $\nabla_a\nabla_b h = f_b E_a(c)+f_a E_b(c)+f\nabla_a\nabla_b E(c) = \nabla_a\nabla_b(fE(c))$. If we take $h^B = fE(c)$, then we see that $S_{ab} - \frac{p}{f}\nabla_a f_b = \rho g_{ab} - \nabla_a h^B$, which means that B is a gradient Ricci soliton.

In [9], the authors proved that

THEOREM 4.3. *Let $M = B \times_f F$ be a gradient Ricci soliton with (h, ρ). If $h_y = 0$ for all y, then F becomes an Einstein space.*

For the converse of theorem 4.3, if we suppose that F is an Einstein space with $\bar{S} = k\bar{g}$ and $h_y = 0$ for all y, then

$$\begin{aligned} \tilde{S}_{ab} &= S_{ab} + \tfrac{p}{f}\nabla_a f_b, \\ \tilde{S}_{ax} &= 0, \\ \tilde{S}_{xy} &= (k - f\triangle f + (p-1)||f_e||^2)\bar{g}_{xy}. \end{aligned} \tag{4.3}$$

Moreover, we can calculate

$$\begin{aligned} \tilde{\nabla}_b\tilde{\nabla}_a h &= \nabla_b\nabla_a h, \\ \tilde{\nabla}_b\tilde{\nabla}_x h &= 0, \\ \tilde{\nabla}_y\tilde{\nabla}_x h &= 0. \end{aligned} \tag{4.4}$$

Hence if we consider the equations (1.2),(4.3) and (4.4), then we can see that

THEOREM 4.4. *Let F be an Einstein space with Einstein constant k and let f be a positive function on B. If there exist some the function h depend only on B and constant ρ satisfying the following equations*

$$\begin{gathered} S_{ab} - \tfrac{p}{f}\nabla_a f_b = \rho g_{ab} - \nabla_a h_b, \\ k - f\triangle f - (p-1)||f_e||^2 = \rho f^2, \end{gathered} \tag{4.5}$$

then the warped product space $B \times_f F$ becomes a gradient Ricci soliton with (h, ρ).

If we use Theorem 4.4, we can construct a model space of a non-Einstein gradient Ricci soliton in the warped product space $M = B\times_f F$.

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References

[1] H. D. Cao, *Geometry of Ricci solitons*, Chinese Ann. Math. Ser. b **27** (2006), 121-142.
[2] A. Derdzinski, *Classification of certain compact . Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*, Math. Z., **172** (1980), 273-280.
[3] M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons : The equation point of view*, Menuscripta Math., **127** (2008), 345-367.
[4] M. Fernández-López and E. García-Río, *A remark on compact Ricci solitons*, Math. Ann., **340** (2008), 893-896.
[5] R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom., **17** (1982), 255-306.
[6] T. Ivey, *Ricci solitons on compact 3-manifolds*, Differential Geom. Appl., **3** (1993), 301-307.
[7] T. Ivey, *New examples of complete Ricci solitons*, Proc. AMS, **122** (1994), 241-245.
[8] B. H. Kim, *Warped products with critical Riemannian metric* , Proc. Japan Academy. **71** (1995), 117-118.
[9] S. D. Lee, B. H. Kim, and J. H. Choi, *On a classification of warped product spaces with gradient Ricci solitons*, Korean J. Math., **24** (2016), 627-636.
[10] G. Perelman, *Ricci flow with surgery on three manifolds*, arXiv:math. DG/0303109 (2003).
[11] P. Petersen and W. Wylie, *On gradient Ricci solitons with symmetry.* Proc. AMS, **137** (2009), 2085-2092.
[12] R. Pina and K. Tenenblat, *On solutions of the Ricci curvature equation and the Einstein equation*, Israel J. Math., **171** (2009), 61-76.

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